TORSION OF AN ANISOTROPIC CURVED BAR

(KRUCHENIE ANIZOTROPNOGO KRIVOGO BRUSA)

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S. G. LEKHNITSKII
(Leningrad)

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Mitchell [1] was the first who presented a rigorous theory for the torsion of a curved bar in the shape of a circular ring segment, analogous to the St. Venant's theory for the prismatic bar. Mitchell considered not only an isotropic case, but also the case of a cylindrical anisotropy and at the same time orthotropy. A group of other authors considered the torsional problem, Gohner [2], Langhaar [3], Solianik-Krassa [4], Rabinovich [5], and others, for the isotropic bars, and Chattarji [6] for the bars having transverse isotropy. The obtained results were either exact or approximate for some particular cross-sections of bars.

The objective of this note is to demonstrate that the exact torsion theory can be extended to the case when a bar exhibits a more complex anisotropy characterized only by the elastic symmetry about any plane of the transverse (radial) cross-section, but otherwise arbitrary.

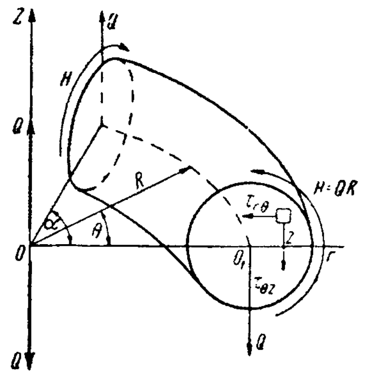
1. Statement of the problem and general equations. Consider a curved bar of an arbitrary cross-section in the shape of a circular ring segment. Denote by R the radius of the center line of the bar, and by a the angle between the end faces. Let the distributed force acting on the lateral faces be reduced to two forces Q and a twisting moment $H = QR$, or equivalent to it, two equal and oppositely directed forces Q acting along the axis of rotation z (see Figure).

We assume that the bar has a curvilinear anisotropy such that every transverse cross-sectional plane (i.e. radial) which is a plane of elastic symmetry is subject to the generalized Hooke's law and undergoes small deflections.

Let the z-axis be the axis of rotation. The generalized Hooke's law relationships in cylindrical coordinates r, θ, z, shown in the figure, are as follows:

$$\begin{aligned}
\varepsilon_r &= a_{11}\sigma_r + a_{12}\sigma_\theta + a_{13}\sigma_z + a_{15}\tau_{rz} \\
\varepsilon_\theta &= a_{12}\sigma_r + a_{22}\sigma_\theta + a_{23}\sigma_z + a_{25}\tau_{rz} \\
\varepsilon_z &= a_{13}\sigma_r + a_{23}\sigma_\theta + a_{33}\sigma_z + a_{35}\tau_{rz} \\
\gamma_{rz} &= a_{15}\sigma_r + a_{25}\sigma_\theta + a_{35}\sigma_z + a_{55}\tau_{rz} \\
\gamma_{\theta z} &= a_{44}\tau_{\theta z} + a_{46}\tau_{r\theta} \\
\gamma_{r\theta} &= a_{46}\tau_{\theta z} + a_{66}\tau_{r\theta}
\end{aligned} \tag{1.1}$$

In the general case the coefficients a_{ik}, thirteen in number, may be functions of the three coordinates. We make only one restriction, viz. that the three coefficients a_{44}, a_{46}, a_{66} are independent of θ, the other ten coefficients being arbitrary. If we supplement the system (1.1) by three equilibrium equations for the continuous medium, we obtain nine equations for the determination of the six stress components and the three displacements components u, v, w along the coordinate directions r, θ and z.



Let us assume that in the given bar, as in the case of an isotropic bar

$$\sigma_r = \sigma_\theta = \sigma_z = \tau_{rz} = 0 \tag{1.2}$$

and consequently $\tau_{\theta z}$ and $\tau_{r\theta}$ are independent of θ. Equations (1.1) will then reduce to

$$\begin{gathered}
\frac{\partial u}{\partial r} = 0, \qquad \frac{\partial v}{\partial \theta} + u = 0, \qquad \frac{\partial w}{\partial z} = 0 \\
\frac{\partial u}{\partial z} + \frac{\partial w}{\partial r} = 0 \\
\frac{1}{r}\frac{\partial w}{\partial \theta} + \frac{\partial v}{\partial z} = a_{44}\tau_{\theta z} + a_{46}\tau_{r\theta}, \qquad \frac{1}{r}\frac{\partial u}{\partial \theta} + \frac{\partial v}{\partial r} - \frac{v}{r} = a_{46}\tau_{\theta z} + a_{66}\tau_{r\theta}
\end{gathered} \tag{1.3}$$

Integrating the first three equations, we express the displacements by three arbitrary functions

$$u = U(\theta, z), \qquad v = V(r, z) - \int U d\theta, \qquad w = W(r, \theta) \tag{1.4}$$

From (1.3) we find general forms of the functions U, W and the relationship between V and the stresses. Thus, the final formulas and equations are

$$u = u' \quad v = v_1(r, z) - \vartheta Rz + v', \quad w = \vartheta R^2\theta + w', \quad v_1 = V + \vartheta Rz \tag{1.5}$$

$$\frac{\partial}{\partial r}(r^2\tau_{r\theta}) + \frac{\partial}{\partial z}(r^2\tau_{\theta z}) = 0$$
$$\frac{\partial}{\partial z}\frac{v_1}{r} = \frac{1}{r}(a_{44}\tau_{\theta z} + a_{46}\tau_{r\theta}) + \vartheta\left(\frac{R}{r} - \frac{R^2}{r^2}\right), \quad \frac{\partial}{\partial r}\frac{v_1}{r} = \frac{1}{r}(a_{46}\tau_{\theta z} + a_{66}\tau_{r\theta}) - \frac{\vartheta Rz}{r^2} \tag{1.6}$$

where u', v' and w' are "rigid" displacements

$$\begin{aligned} u' &= z(\omega_2\cos\theta - \omega_1\sin\theta) + a\cos\theta + b\sin\theta \\ v' &= -z(\omega_2\sin\theta + \omega_1\cos\theta) - a\sin\theta + b\cos\theta + \omega_3 r \\ w' &= -r(\omega_2\cos\theta - \omega_1\sin\theta) + c \end{aligned} \tag{1.7}$$

The constants ϑ, ω_i, a, b, c are to be determined.

The stresses on the lateral surface have to satisfy the following condition:

$$\tau_{r\theta}\cos(n, r) + \tau_{\theta z}\cos(n, z) = 0 \tag{1.8}$$

On the end faces, as well as in any transverse cross-section, there must be satisfied the equilibrium conditions

$$\iint \tau_{r\theta}\, dr\, dz = 0, \qquad \iint \tau_{\theta z}\, dz\, dz = Q$$
$$\iint [\tau_{r\theta}\, z - \tau_{\theta z}(r - R)]\, dr\, dz = H = QR \tag{1.9}$$

where the integrals are taken over the whole area of the transverse cross-section.

It is seen from (1.5) to (1.7) that the displacements do not contain elastic moduli explicitly, i.e. in their form these equations do not differ from equations for the displacements of isotropic bars [5]. The influence of the kind of anisotropy considered here is shown in that the function v_1 is determined from more complex equations (1.6) and the constant ϑ is different.

2. Determination of the stress function, the constant and the displacements. The further development of the solution of the problem is the same as in the isotropic case, (see, e.g. [5]). We introduce the stress function $F(r, z)$ setting

$$\tau_{\theta z} = \frac{R^2}{r^2}\frac{\partial F}{\partial r}, \qquad \tau_{r\theta} = -\frac{R^2}{r^2}\frac{\partial F}{\partial z} \tag{2.1}$$

Eliminating v_1 from (1.7) we obtain the following equation for F:

$$\frac{\partial}{\partial r}\left[\frac{1}{r^3}\left(a_{44}\frac{\partial F}{\partial r} - a_{46}\frac{\partial F}{\partial z}\right)\right] - \frac{\partial}{\partial z}\left[\frac{1}{r^3}\left(a_{46}\frac{\partial F}{\partial r} - a_{66}\frac{\partial F}{\partial z}\right)\right] = -\frac{2\vartheta}{r^3} \tag{2.2}$$

The problem thus reduces to a determination in the region of the transverse cross-section of a function F which satisfies (2.2) and which is constant on the contour of the cross-section. Function F is determined within a multiplicative constant ϑ which is found from (1.9).

In the case of a simply-connected region, it is possible to assume that $F = 0$ on the contour. All equations of (1.9) are thus reduced to

$$2R^3 \iint \frac{F}{r^3}\, dr\, dz = H \tag{2.3}$$

In order to determine six arbitrary constants ω_i, a, b, c it is necessary, in addition, to specify the end conditions. We shall consider one of the possible versions. We shall assume that the right end of the bar is clamped in such a way that the center of gravity of the end cross-section and the linear element associated with it directed along the axis of the bar and a plane element in the $r\theta$-plane remain fixed. We have:

$$u = v = w = 0, \quad \frac{\partial u}{\partial \theta} = \frac{\partial w}{\partial \theta} = \frac{\partial w}{\partial r} = 0 \quad \text{при} \begin{cases} r = R \\ \theta = z = 0 \end{cases} \tag{2.4}$$

The displacements are:

$$u = \vartheta R z \sin\theta, \quad v = v_1(r, z) - v_1(R, 0)\frac{r}{R} - \vartheta R z(1 - \cos\theta) \tag{2.5}$$
$$w = \vartheta R(R\theta - r\sin\theta)$$

The component of rotation about the axis tangent to the axis of the bar $r = R$ is

$$\omega_\theta = \vartheta R \sin\theta \tag{2.6}$$

From (2.3) we obtain

$$\vartheta = \frac{H}{C}, \qquad C = \frac{2R^3}{\vartheta}\iint \frac{F}{r^3}\,dr\,dz \tag{2.7}$$

The bending of the left end is

$$f = \frac{HR^2}{C}(\alpha - \sin\alpha) \tag{2.8}$$

With the increasing R the expression (2.5) tends to the expressions for the displacements of the twisted straight bar. Rabinovich suggested, analogously to the prismatic bar, that D and C be referred to as the "angle of twist" and "torsional rigidity", respectively [5].

3. Homogeneous bar with cylindrical anisotropy. In the case of a homogeneous bar with cylindrical anisotropy, with the axis of anisotropy directed along the axis of rotation, all coefficients a_{ik} in (1.1) are constants. We introduce new variables

$$\rho = r,\ \zeta = \frac{z + mr}{\sqrt{n^2 - m^2}} \qquad \left(m = \frac{a_{46}}{a_{44}},\ n = \sqrt{\frac{a_{66}}{a_{44}}} = \sqrt{\frac{G_{\theta z}}{G_{r\theta}}}\right) \tag{3.1}$$

where $G_{\theta z}$, $G_{r\theta}$ are the shear moduli which characterize the changes of the angles between the r and θ directions in the planes normal to the axis caused by the stresses $\tau_{\theta z}$ and $\tau_{r\theta}$, respectively. With these new coordinates Equation (2.2) will be transformed into the equation for the isotropic bar with the shear modulus $G_{\theta z}$ and "angle of twist"

$$\frac{\partial^2 F}{\partial \rho^2} + \frac{\partial^2 F}{\partial \zeta^2} - \frac{3}{\rho}\frac{\partial F}{\partial \rho} = -2\vartheta G_{\theta z} \tag{3.2}$$

The problem is thus reduced to the torsion problem of an isotropic bar of the same radius R, and with the area of the cross-section obtained from a given one by means of the affine transformation (3.1), since on the contour of the mapped region (simply connected) F also must be zero.

From the stresses at a point ρ, ζ of an isotropic bar (with an accuracy within a multiplicative constant $G_{\theta z}$) we can determine the stresses $\tau_{\theta z}$, $\tau_{r\theta}$ at a corresponding point of an anisotropic bar

$$\tau_{\theta z} = \tau_{\theta\zeta} - \frac{m}{\sqrt{n^2 - m^2}}\tau_{\rho\theta}, \qquad \tau_{r\theta} = \frac{1}{\sqrt{n^2 - m^2}}\tau_{\rho\theta} \tag{3.3}$$

The constant ϑ for a given moment H can be found from (2.7). The rigidity of the anisotropic bar connected with the rigidity of an auxiliary isotropic bar C_0 by means of (2.7) and (3.1) has, in this case, a simple expression

$$C = \sqrt{n^2 - m^2}\, C_0 \tag{3.4}$$

In the case of an orthotropic material $a_{46} = m = 0$.

In cases where sufficiently accurate data exist for isotropic bars, Equations (3.3) and (3.4) permit an easy application of these data for the anisotropic cases, thus avoiding cumbersome manipulations and computations.

Consider, as an example, an orthotropic bar of a rectangular cross-section. Denote by R, b, h its radius, and the width and height of its cross-section, and by R, b_1, h_1, the same magnitudes for the auxiliary isotropic case. In [5] all necessary formulas and numerical data are given for the isotropic rectangular bars. Torsional rigidity and stresses at the mid-points of the h_1-sides closest to the z-axis are determined from the formulas

$$C_0 = G_{\theta z}\, b_1 h_1^3\, k\left(\frac{2R}{b_1},\ \frac{b_1}{h_1}\right), \qquad (\tau_{\theta\zeta})_0 = G_{\theta z}\, \vartheta h_1 k_1\left(\frac{2R}{b_1},\ \frac{b_1}{h_1}\right) \tag{3.5}$$

The values of the coefficients k, k_1 for a series of ratios $2R/b_1$, b_1/h_1 which are used here, are presented in Tables 1 and 2 of [5].

For the calculation of the torsional rigidity and stresses in an orthotropic bar for which

$$\frac{b}{h} = \alpha, \qquad \frac{G_{\theta z}}{G_{r\theta}} = n^2 \tag{3.6}$$

we must consider an isotropic bar with the same radius R with the dimensions $b_1 = b$, $h_1 = h/n$, $b_1/h_1 = n$. From (3.3), (3.4) and (3.5) we obtain

$$C = G_{\theta z}\, h^4 \frac{\alpha}{n^2}\, k\left(\frac{2R}{b},\ n\alpha\right), \qquad (\tau_{\theta z})_0 = \frac{H}{h^3}\frac{n}{\alpha}\, \frac{k_1\left(\frac{2R}{b},\ n\alpha\right)}{k\left(\frac{2R}{b},\ n\alpha\right)} \tag{3.7}$$

For the sake of definiteness, let us take $2R/b = 3$; $\alpha = 3$ and determine formulas for the two relationships between the shear moduli.

Case 1. $G_{\theta z} = 4G_{r\theta}$ ($G_{\theta z} > G_{r\theta}$), $n = 2$.

$$C = G_{\theta z}\, h^4\, 0.75\, k\,(3,6), \qquad (\tau_{\theta z})_0 = \frac{H}{h^3}\, \frac{2k_1(3,\ 6)}{3k\,(3,\ 6)} \tag{3.8}$$

In [5] the values of k (3.6) and k_1 (3.6) are not shown. They can be found approximately from tables presented in [5] by means of linear interpolation. Thus, we obtain k (3.6) = 0.355, k_1 (0.36 = 1.53

$$C = G_{\theta z} h^4 0.266, \qquad (\tau_{\theta z})_0 = \frac{H}{h^3} 2.87 \tag{3.9}$$

Case 2. $G_{\theta z} = 0.25\, G_{r\theta}$ $(G_{\theta z} < G_{r\theta})$; $n = 0.5$

$$C = G_{\theta z} h^4 12\, k\,(3,1.5) = G_{\theta z} h^4 2.544, \qquad (\tau_{\theta z})_0 = \frac{H}{h^3} 0.969 \tag{3.10}$$

For the isotropic bar with $2R/b = 3$; $a = 3$, we have

$$C = G_{\theta z} h^4 0.912, \qquad (\tau_{\theta z})_0 = \frac{H}{h^3} 1.556 \tag{3.11}$$

Comparing all these data (and the results calculated for other dimensions not presented here) we observe that for a given $G_{\theta z}$ the torsional rigidity is decreasing with increasing ratio of the shear moduli, and vice-versa. The behavior of the stress $(\tau_{\theta z})_0$ is reversed; for a given twisting moment the stress is increasing with increasing ratio $G_{\theta z}/G_{r\theta}$. The isotropic case falls, as can be seen from (3.9) through (3.11), between Cases 1 and 2.

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